New exact double periodic wave and complex wave solutions for a generalized sinh–Gordon equation

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Elliptic equation
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Exact solution

Abstract
In this paper, dependent and independent variable transformations are introduced to solve a generalized sinh–Gordon equation by using the binary F-expansion method and the knowledge of elliptic equation and Jacobian elliptic functions. Many different new exact solutions such as double periodic wave and complex wave solutions are obtained. Some previous results are extended.

1. Introduction
It is well known that the exact solutions of the sinh–Gordon equations have been extensively studied in the field of theoretical physics (see Refs. [1–7] and references cited therein).

In 2006, Wazwaz [8] studied the following generalized Sinh–Gordon equation:

\[ u_{tt} - au_{xx} + b \sinh(nu) = 0, \]  

where \( n \) is a positive integer and \( a, b \) are two constants. And he derived families of exact solutions using the reliable tanh method. Tang et al. [9] studied the bifurcation behaviors and exact solutions of the Eq. (1) under three different functions transformations by using the bifurcation theory of dynamical system.

In this paper, we aim to extend the previous works in Refs. [8,9], we shall obtain many new exact solutions of Eq. (1), including double periodic wave and complex wave solutions.

This paper is organized as follows. In Section 2, we introduce the binary F-expansion method briefly. In Section 3, we give many exact solutions of Eq. (1). In Section 4, a short conclusion will be given.

2. The binary F-expansion method
For a given nonlinear partial differential equation

\[ \Phi(f(u), u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0, \]  

where \( f(u) \) is a composite function which is similar to \( \sin(nu) \) or \( \sinh(nu) \) \((n = 1, 2, \ldots)\) etc. As in Ref. [15], the binary F-expansion method is simply represented as follows:

**Step 1:** We make a transformation

\[ u = \phi \left( \frac{U(\eta)}{V(\eta)} \right), \]
where $\xi = \lambda_1(x + c_1 t)$, $\eta = \lambda_2(x + c_2 t)$, $\lambda_1, \lambda_2, c_1, c_2$ are unknown parameters which to be further determined. The transformation $u = \sqrt[4]{\frac{u''(\xi)}{u''(\eta)}}$ was first given by Lamb and used it to solve the sine–Gordon equation [12], $u = \tan^{-1} \left( \frac{u''(\xi)}{u''(\eta)} \right)$ and $u = \sqrt[4]{\tan^{-1} \left( \frac{u''(\xi)}{u''(\eta)} \right)}$ are its two special cases. Substituting (3) into (2), yields

$$\Phi(U, U', U'', \ldots, V, V', V'', \ldots) = 0.$$ \hspace{1cm} (4)

**Step 2:** On some constraint conditions, if Eq. (4) can be differentiated as follows

$$U'^2 = P_1 + Q_1 U^2 + R_1 U^4,$$

$$V'^2 = P_2 + Q_2 V^2 + R_2 V^4,$$ \hspace{1cm} (5) (6)

where $P_1, Q_1, R_1, P_2, Q_2, R_2$ are some parameters, then, with the aid of Table 1 (see Ref. [14]), we can get the solutions $U(\xi), V(\eta)$ of Eqs. (5) and (6).

**Step 3:** Substituting the $U(\xi), V(\eta)$ into (3), many exact solutions of Eq. (2) can be obtained.

### 3. Exact solutions of Eq. (1)

First, let us recall some properties of Jacobian elliptic functions. We know that there exist twelve kinds of Jacobian elliptic functions [10,11]

$$sn(\tau, m), cn(\tau, m), dn(\tau, m), sc(\tau, m), sd(\tau, m), cd(\tau, m),$$

$$ns(\tau, m), nc(\tau, m), nd(\tau, m), cs(\tau, m), ds(\tau, m), dc(\tau, m),$$

where $m (0 < m < 1)$ is a modulus of Jacobian elliptic functions.

When $m \to 1$, the Jacobian functions degenerate to the hyperbolic functions, that is

$$sn(\tau, m) \to \tanh(\tau), cn(\tau, m) \to \sech(\tau), dn(\tau, m) \to \sech(\tau), sc(\tau, m) \to \sinh(\tau),$$

$$sd(\tau, m) \to \sinh(\tau), cd(\tau, m) \to 1, ns(\tau, m) \to \coth(\tau), nc(\tau, m) \to \cosh(\tau),$$

$$nd(\tau, m) \to \cosh(\tau), cs(\tau, m) \to \csc(\tau), ds(\tau, m) \to \csc(\tau), dc(\tau, m) \to 1.$$ \hspace{1cm} (7)

When $m \to 0$, the Jacobian functions degenerate to the trigonometric functions, i.e.

$$sn(\tau, m) \to \sin(\tau), cn(\tau, m) \to \cos(\tau), dn(\tau, m) \to 1, \quad sc(\tau, m) \to \tan(\tau),$$

$$sd(\tau, m) \to \sin(\tau), cd(\tau, m) \to \cos(\tau), ns(\tau, m) \to \csc(\tau), nd(\tau, m) \to 1, cs(\tau, m) \to \cot(\tau),$$

$$dc(\tau, m) \to \sec(\tau), ds(\tau, m) \to \csc(\tau), dc(\tau, m) \to \sec(\tau).$$

Next, we study Eq. (1). Considering the following transformation:

$$\xi = \lambda_1(x + ct), \quad \eta = \lambda_2(x + \frac{a}{c} t), \quad a \neq c^2,$$ \hspace{1cm} (8)

where $\lambda_1, \lambda_2$ are two parameters to be determined later, Eq. (1) can be rewritten as

$$\lambda_1^2 (c^2 - a)u_{\xi\xi} + \lambda_2^2 (a - c^2)u_{\eta\eta} + bc^2 \sinh(nu) = 0.$$ \hspace{1cm} (9)

By means of a similar ansatz as given in Refs. [12,13], letting

<table>
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<tr>
<th>$P$</th>
<th>$Q$</th>
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<td>$sn(\tau, m), \pm i cs(\tau, m), \pm i = -1$</td>
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</table>
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{U(\xi)}{V(\eta)} \right) \]  

(9) for Eq. (8), yields

\[ i \lambda^2 \left( c^2 - a \right) \left( U^2 - V^2 \right) \frac{U_{\xi\xi}}{U} - 2U_x^2 + i \lambda^2 a \left( a - c^2 \right) \left( V^2 - U^2 \right) \frac{V_{\eta\eta}}{V} - 2V_x^2 = nbc^2 \left(U^2 + V^2\right). \]  

(10)

Successive differentiation of this result with respect to both \( \xi \) and \( \eta \) results in

\[ \frac{c^2}{UU_{\xi}} \frac{U_{\xi\xi}}{U} = a V \frac{V_{\eta\eta}}{V} = 0, \]  

(11)

and from (11), one has

\[ \frac{c^2}{UU_{\xi}} \frac{U_{\xi\xi}}{U} = a V \frac{V_{\eta\eta}}{V} = \omega, \]  

(12)

where \( \omega \) is a parameter to be determined later, i.e.

\[ U_x^2 = \frac{\omega}{4c^2} U^4 + \mu_1 U^2 + v_1, \quad V_x^2 = \frac{\omega}{4a} V^4 + \mu_2 V^2 + v_2, \]  

(13)

where \( \mu_1, \ v_1, \ \mu_2, \ v_2 \) are integral constants.

Considering (10) and (13), we have the corresponding constraint conditions

\[ i \lambda^2 \left( c^2 - a \right) \mu_1 + i \lambda^2 a \left( a - c^2 \right) \mu_2 + nbc^2 = 0, \quad c^2 v_1 - av_2 = 0. \]  

(14)

Obviously, not all Jacobi elliptic functions satisfying (13) can satisfy the constraint conditions (14). Only some combinations of these Jacobi elliptic functions are the solutions that the Eq. (1) can admit. With the aid of Table 1, we will show the details. There are 31 cases which need to be addressed.

**Case 1.** From Table 1, choosing \( U = sn(\xi, k) \) (or \( U = cd(\xi, k) \)) and \( V = nd(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = k^2, \quad \mu_1 = -(1 - k^2), \quad v_1 = 1, \quad \frac{\omega}{4a} = m^2 - 1, \quad \mu_2 = 2 - m^2, \quad v_2 = -1. \]  

(15)

Substituting (15) into the constraint conditions (14), the parameters can be determined as

\[ k = \sqrt{1 - m^2}, \quad \omega = 4a(m^2 - 1), \quad \lambda^2 = \frac{nb}{4a(1 - m^2)}, \quad c^2 = -a, \]  

(16)

then the double periodic wave solutions to the Eq. (1) are

\[ u = \frac{4}{n} \tanh^{-1}(sn(\xi, k)dn(\eta, m)), \]  

(17)

and

\[ u = \frac{4}{n} \tanh^{-1}(cd(\xi, k)dn(\eta, m)). \]  

(18)

The profile of (17) is shown in Fig. 1(1-1).

**Case 2.** From Table 1, choosing \( U = sn(\xi, k) \) (or \( U = cd(\xi, k) \)) and \( V = ns(\eta, m) \pm cs(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = k^2, \quad \mu_1 = -(1 - k^2), \quad v_1 = 1, \quad \frac{\omega}{4a} = \frac{1}{4}, \quad \mu_2 = \frac{1 - 2m^2}{2}, \quad v_2 = \frac{1}{4}. \]  

(19)

Substituting (19) into the constraint conditions (14), the parameters can be determined as

\[ k = 1, \quad \omega = a, \quad \lambda^2 = \frac{2nb}{3a(2m^2 - 1)}, \quad c^2 = \frac{a}{4}, \]  

(20)

then the complex wave solutions to the Eq. (1) are

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\tanh(\xi)}{ns(\eta, m) \pm cs(\eta, m)} \right). \]  

(21)
and
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{1}{\text{ns}(\eta, m) \pm \text{cs}(\eta, m)} \right). \]  

(22)

The profiles of (21) are shown in Fig. 1(1–2) and (1–3).

When \( m \to 0 \), the solutions (21), (22) turn to be another complex wave and periodic wave solutions respectively

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\tanh(\zeta)}{\csc(\eta) \pm \cot(\eta)} \right), \]  

(23)

and

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{1}{\csc(\eta) \pm \cot(\eta)} \right). \]  

(24)

Case 3. From Table 1, choosing \( U = \text{sn}(\xi, k) \) (or \( U = \text{cd}(\xi, k) \)) and \( V = \text{nc}(\eta, m) \pm \text{sc}(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = k^2, \quad \mu_1 = -(1 - k^2), \quad \nu_1 = 1, \quad \frac{\omega}{4a} = \frac{1 - m^2}{4}, \quad \mu_2 = \frac{1 + m^2}{2}, \quad \nu_2 = \frac{1 - m^2}{4}. \]  

(25)
Substituting (25) into the constraint conditions (14), the parameters can be determined as
\[ k = 1, \quad \omega = a(1 - m^2), \quad \lambda^2 = \frac{2nb(m^2 - 1)}{a(m^2 + 1)(m^2 + 3)}, \quad c^2 = \frac{a(1 - m^2)}{4}, \] (26)
then the complex wave solutions to the Eq. (1) are
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\tanh(\xi)}{nc(\eta, m) \pm sc(\eta, m)} \right), \] (27)
and
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{1}{nc(\eta, m) \pm sc(\eta, m)} \right), \] (28)
when \( m \to 0 \), the solutions (27), (28) turn to be another complex wave and periodic wave solutions respectively
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\tanh(\xi)}{\sec(\eta) \pm \tan(\eta)} \right), \] (29)
and
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{1}{\sec(\eta) \pm \tan(\eta)} \right). \] (30)

**Case 4.** From Table 1, choosing \( U = sn(\xi, k) \) (or \( U = cd(\xi, k) \)) and \( V = ns(\eta, m) \pm ds(\eta, m) \), respectively, and then from (13), we have
\[ \frac{\omega}{4cz} = k^2, \quad \mu_1 = -(1 - k^2), \quad \nu_1 = 1, \quad \frac{\omega}{4a} = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}. \] (31)
Substituting (31) into the constraint conditions (14), the parameters can be determined as
\[ k = 1, \quad m = 1, \quad \omega = a, \quad \lambda^2 = \frac{2nb}{3a}, \quad c^2 = \frac{a}{4}. \] (32)
then the complex wave solutions to the Eq. (1) are
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\tanh(\xi)}{\coth(\eta) \pm \csch(\eta)} \right), \] (33)
and
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{1}{\coth(\eta) \pm \csch(\eta)} \right). \] (34)

**Case 5.** From Table 1, choosing \( U = sn(\xi, k) \) (or \( U = cd(\xi, k) \)) and \( V = sn(\eta, m) \pm ics(\eta, m) \), \( i^2 = -1 \), respectively, and then from (13), we have
\[ \frac{\omega}{4cz} = k^2, \quad \mu_1 = -(1 - k^2), \quad \nu_1 = 1, \quad \frac{\omega}{4a} = \frac{m^2}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}. \] (35)
Substituting (35) into the constraint conditions (14), the parameters can be determined as
\[ k = 1, \quad \omega = am^2, \quad \lambda^2 = \frac{2m^2nb}{a(m^2 - 2)(m^2 - 4)}, \quad c^2 = \frac{m^2a}{4}. \] (36)
then the complex wave solutions to the Eq. (1) are
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\tanh(\xi)}{sn(\eta, m) \pm ics(\eta, m)} \right), \] (37)
and
\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{1}{sn(\eta, m) \pm ics(\eta, m)} \right), \] (38)
when \( m \to 1 \), the solutions (37), (38) turn to be another complex wave solutions respectively
\[ u = \frac{4}{n} \tanh^{-1}\left( \frac{\tanh(\xi)}{\tanh(\eta) \pm \text{csch}(\eta)} \right), \]  

(39)

and

\[ u = \frac{4}{n} \tanh^{-1}\left( \frac{1}{\tanh(\eta) \pm \text{csch}(\eta)} \right). \]  

(40)

**Case 6.** From Table 1, choosing \( U = cn(\xi, k) \) and \( V = sd(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = -k^2, \quad \mu_1 = 2k^2 - 1, \quad \nu_1 = 1 - k^2, \quad \frac{\omega}{4a} = -m^2(1 - m^2), \quad \mu_2 = 2m^2 - 1, \quad \nu_2 = 1. \]  

(41)

Substituting (41) into the constraint conditions (14), the parameters can be determined as

\[ k^2 = \frac{m^2(1 - m^2)}{1 + m^2 - m^4}, \quad \omega = 4am^2(2m^2 - 1), \quad \lambda^2 = \frac{nb(1 + m^2 - m^4)}{am^2(2 + 3m^2 - m^6)}, \quad \epsilon^2 = \alpha(1 + m^2 - m^4), \]  

then the double periodic wave solution to the Eq. (1) is

\[ u = \frac{4}{n} \tanh^{-1}(cn(\xi, k)sd(\eta, m)). \]  

(43)

**Case 7.** From Table 1, choosing \( U = cn(\xi, k) \) and \( V = ds(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = -k^2, \quad \mu_1 = 2k^2 - 1, \quad \nu_1 = 1 - k^2, \quad \frac{\omega}{4a} = 1, \quad \mu_2 = 2m^2 - 1, \quad \nu_2 = -m^2(1 - m^2). \]  

(44)

Substituting (44) into the constraint conditions (14), the parameters can be determined as

\[ k^2 = \frac{1}{1 + m^2 - m^4}, \quad \omega = 4a, \quad \lambda^2 = -a(1 + m^2 - m^4), \quad \epsilon^2 = \frac{nb(1 + m^2 - m^4)}{am^2(2 + 3m^2 - m^6)}, \]  

then the double periodic wave solution to the Eq. (1) is

\[ u = \frac{4}{n} \tanh^{-1}(cn(\xi, k)sd(\eta, m)). \]  

(46)

**Case 8.** From Table 1, choosing \( U = cn(\xi, k) \) and \( V = ns(\eta, m) + ds(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = -k^2, \quad \mu_1 = 2k^2 - 1, \quad \nu_1 = 1 - k^2, \quad \frac{\omega}{4a} = \frac{1}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}. \]  

(47)

Substituting (47) into the constraint conditions (14), the parameters can be determined as

\[ k = 1, \quad m = 0, \quad \omega = a, \quad \lambda^2 = -\frac{4nb}{15a}, \quad \epsilon^2 = -\frac{a}{4}, \]  

then the complex wave solution to the Eq. (1) is

\[ u = \frac{4}{n} \tanh^{-1}\left( \frac{1}{2} \text{sech}(\xi) \sin(\eta) \right). \]  

(49)

The profile of (49) is shown in Fig. 1(1–4).

**Case 9.** From Table 1, choosing \( U = dn(\xi, k) \) and \( V = cs(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = -1, \quad \mu_1 = 2 - k^2, \quad \nu_1 = k^2 - 1, \quad \frac{\omega}{4a} = 1, \quad \mu_2 = 2 - m^2, \quad \nu_2 = 1 - m^2. \]  

(50)

Substituting (50) into the constraint conditions (14), the parameters can be determined as

\[ k = m, \quad \omega = 4a, \quad \lambda^2 = \frac{nb}{4a(2 - m^2)}, \quad \epsilon^2 = -a, \]  

then the double periodic wave solution to the Eq. (1) is

\[ u = \frac{4}{n} \tanh^{-1}(dn(\xi, m)sc(\eta, m)). \]  

(52)
when $m \to 0$, the solution (52) turn to be another periodic wave solution

$$u = \frac{4}{n} \tanh^{-1}(\tan(\eta)).$$

**Case 10.** From Table 1, choosing $U = dn(\xi, k)$ and $V = ns(\eta, m) \pm ds(\eta, m)$, respectively, and then from (13), we have

$$\frac{\omega}{4c^2} = -1, \quad \mu_1 = 2 - k^2, \quad \nu_1 = k^2 - 1, \quad \frac{\omega}{4a} = \frac{1}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}.$$

Substituting (54) into the constraint conditions (14), the parameters can be determined as

$$k = \sqrt{1 - m^2}, \quad \omega = a, \quad \lambda^2 = \frac{4nb}{15a(m^2 - 1)}, \quad c^2 = -\frac{a}{4}.$$

then the complex wave solution to the Eq. (1) is

$$u = \frac{4}{n} \tanh^{-1}\left(\frac{dn(\xi, k)}{ns(\eta, m) \pm ds(\eta, m)}\right).$$

**Case 11.** From Table 1, choosing $U = ns(\xi, k)$ and $V = ns(\eta, m) \pm cs(\eta, m)$, respectively, and then from (13), we have

$$\frac{\omega}{4c^2} = 1, \quad \mu_1 = -(1 + k^2), \quad \nu_1 = k^2, \quad \frac{\omega}{4a} = \frac{1}{4}, \quad \mu_2 = \frac{1 - 2m^2}{2}, \quad \nu_2 = \frac{1}{4}.$$

Substituting (57) into the constraint conditions (14), the parameters can be determined as

$$k = 1, \quad \omega = a, \quad \lambda^2 = \frac{nb}{3a(m^2 - 1)}, \quad c^2 = \frac{a}{4}.$$

then the complex wave solution to the Eq. (1) is

$$u = \frac{4}{n} \tanh^{-1}\left(\frac{\coth(\xi)}{ns(\eta, m) \pm cs(\eta, m)}\right).$$

when $m \to 0$, the solution (59) turn to be another complex wave solution

$$u = \frac{4}{n} \tanh^{-1}\left(\frac{\coth(\xi)}{\csc(\eta) \pm \cot(\eta)}\right).$$

**Case 12.** From Table 1, choosing $U = ns(\xi, k)$ and $V = nc(\eta, m) \pm sc(\eta, m)$, respectively, and then from (13), we have

$$\frac{\omega}{4c^2} = 1, \quad \mu_1 = -(1 + k^2), \quad \nu_1 = k^2, \quad \frac{\omega}{4a} = \frac{1 - m^2}{4}, \quad \mu_2 = \frac{1 + m^2}{2}, \quad \nu_2 = \frac{1 - m^2}{4}.$$

Substituting (61) into the constraint conditions (14), the parameters can be determined as

$$k = 1, \quad \omega = a(1 - m^2), \quad \lambda^2 = \frac{nb(m^2 - 1)}{3 + m^2}, \quad c^2 = \frac{a(1 - m^2)}{4}.$$

then the complex wave solution to the Eq. (1) is

$$u = \frac{4}{n} \tanh^{-1}\left(\frac{\coth(\xi)}{nc(\eta, m) \pm sc(\eta, m)}\right).$$

when $m \to 0$, the solution (63) turn to be another complex wave solution

$$u = \frac{4}{n} \tanh^{-1}\left(\frac{\coth(\xi)}{\sec(\eta) \pm \tan(\eta)}\right).$$

**Case 13.** From Table 1, choosing $U = ns(\xi, k)$ (or $U = dc(\xi, k)$) and $V = ns(\eta, m) \pm ds(\eta, m)$, respectively, and then from (13), we have

$$\frac{\omega}{4c^2} = 1, \quad \mu_1 = -(1 + k^2), \quad \nu_1 = k^2, \quad \frac{\omega}{4a} = \frac{1}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}.$$

Substituting (65) into the constraint conditions (14), the parameters can be determined as
\[ k = m, \quad \omega = a, \quad \lambda^2 = \frac{4nb}{9a(1 - m^2)}, \quad c^2 = \frac{a}{4}, \]

then the complex wave solutions to the Eq. (1) are

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{ns(\xi, k)}{ns(\eta, m) \pm ds(\eta, m)} \right) \]

and

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{dc(\xi, k)}{ns(\eta, m) \pm ds(\eta, m)} \right). \]

When \( m \to 0 \), the solutions (67) and (68) turn to be another double periodic wave solutions respectively

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{1}{2} \csc(\xi) \sin(\eta) \right). \]

\[ \text{Case 14.} \] From Table 1, choosing \( U = ns(\xi, k) \) and \( V = sn(\eta, m) \pm ics(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = 1, \quad \mu_1 = -(1 + k^2), \quad v_1 = k^2, \quad \frac{\omega}{4a} = \frac{m^2}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad v_2 = \frac{m^2}{4}. \]

Substituting (71) into the constraint conditions (14), the parameters can be determined as

\[ k = 1, \quad \omega = am^2, \quad \lambda^2 = \frac{m^2nb}{a(1 - m^2)(4 - m^2)}, \quad c^2 = \frac{am^2}{4}. \]

then the complex wave solution to the Eq. (1) is

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\coth(\xi)}{sn(\eta, m) \pm ics(\eta, m)} \right). \]

\[ \text{Case 15.} \] From Table 1, choosing \( U = nc(\xi, k) \) and \( V = sc(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = 1 - k^2, \quad \mu_1 = 2k^2 - 1, \quad v_1 = -k^2, \quad \frac{\omega}{4a} = 1 - m^2, \quad \mu_2 = 2 - m^2, \quad v_2 = 1. \]

Substituting (74) into the constraint conditions (14), the parameters can be determined as

\[ k = 1, \quad m = 1, \quad \omega = 0, \quad \lambda^2 = \frac{nb}{4a}, \quad c^2 = -a. \]

then the complex wave solution to the Eq. (1) is

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\cosh(\xi)}{\sinh(\eta)} \right). \]

The profile of (76) is shown in Fig. 1(1–6).

\[ \text{Case 16.} \] From Table 1, choosing \( U = nc(\xi, k) \) and \( V = sd(\eta, m) \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = 1 - k^2, \quad \mu_1 = 2k^2 - 1, \quad v_1 = -k^2, \quad \frac{\omega}{4a} = -m^2(1 - m^2), \quad \mu_2 = 2m^2 - 1, \quad v_2 = 1. \]

Substituting (77) into the constraint conditions (14), the parameters can be determined as

\[ k = \frac{1}{\sqrt{1 + m^2 - m^4}}, \quad \omega = 4am^2(m^2 - 1), \quad \lambda^2 = \frac{nb(1 + m^2 - m^4)}{m^2a(2 + 3m^2 - m^6)}, \quad c^2 = a(m^4 - m^2 - 1). \]

then the double periodic wave solution to the Eq. (1) is

\[ u = \frac{4}{n} \tanh^{-1} (nc(\xi, k)ds(\eta, m)). \]
Case 17. From Table 1, choosing $U = nc(\xi, k)$ and $V = ds(\eta, m)$, respectively, and then from (13), we have

$$\frac{\omega}{4c^2} = 1 - k^2, \quad \mu_1 = 2k^2 - 1, \quad v_1 = -k^2, \quad \frac{\omega}{4a} = 1, \quad \mu_2 = 2m^2 - 1, \quad v_2 = -m^2(1 - m^2). \quad (80)$$

Substituting (80) into the constraint conditions (14), the parameters can be determined as

$$k = \frac{m\sqrt{(1 - m^2)(1 + m^2 - m^4)}}{1 + m^2 - m^4}, \quad \omega = 4a, \quad \lambda^2 = \frac{nb(1 + m^2 - m^4)}{am^2(1 - m^4)}, \quad c^2 = a(1 + m^2 - m^4). \quad (81)$$

then the double periodic wave solution to the Eq. (1) is

$$u = \frac{4}{n}\tanh^{-1}(nc(\xi, k)sd(\eta, m)). \quad (82)$$

Case 18. From Table 1, choosing $U = nd(\xi, k)$ and $V = sc(\eta, m)$, respectively, and then from (13), we have

$$\frac{\omega}{4c^2} = k^2 - 1, \quad \mu_1 = 2 - k^2, \quad v_1 = -1, \quad \frac{\omega}{4a} = 1 - m^2, \quad \mu_2 = 2 - m^2, \quad v_2 = 1. \quad (83)$$

Substituting (83) into the constraint conditions (14), the parameters can be determined as

$$k = m, \quad \omega = 4a(1 - m^2), \quad \lambda^2 = \frac{nb}{4a(2 - m^2)}, \quad c^2 = -a. \quad (84)$$

then the double periodic wave solution to the Eq. (1) is

$$u = \frac{4}{n}\tanh^{-1}(nd(\xi, k)cs(\eta, m)), \quad (85)$$

when $m \to 0$, the solution (85) turn to be another periodic wave solution

$$u = \pm \frac{4}{n}\tanh^{-1}(\cot(\eta)). \quad (86)$$

Case 19. From Table 1, choosing $U = sc(\xi, k)$ and $V = ns(\eta, m) \pm cs(\eta, m)$, respectively, and then from (13), we have

$$\frac{\omega}{4c^2} = 1 - k^2, \quad \mu_1 = 2 - k^2, \quad v_1 = 1, \quad \frac{\omega}{4a} = \frac{1}{4}, \quad \mu_2 = \frac{1 - 2m^2}{2}, \quad v_2 = \frac{1}{4}. \quad (87)$$

Substituting (87) into the constraint conditions (14), the parameters can be determined as

$$k = 0, \quad \omega = a, \quad \lambda^2 = \frac{nb}{3am^2}, \quad c^2 = \frac{a}{4}. \quad (88)$$

then the complex wave solution to the Eq. (1) is

$$u = \frac{4}{n}\tanh^{-1}\left(\frac{\tan(\xi)}{ns(\eta, m) \pm cs(\eta, m)}\right). \quad (89)$$

when $m \to 1$, the solution (89) turn to be another complex wave solution

$$u = \frac{4}{n}\tanh^{-1}\left(\frac{\tan(\xi)}{\cot(\eta) \pm csch(\eta)}\right). \quad (90)$$

Case 20. From Table 1, choosing $U = sc(\xi, k)$ and $V = nc(\eta, m) \pm sc(\eta, m)$, respectively, and then from (13), we have

$$\frac{\omega}{4c^2} = 1 - k^2, \quad \mu_1 = 2 - k^2, \quad v_1 = 1, \quad \frac{\omega}{4a} = \frac{1 - m^2}{4}, \quad \mu_2 = \frac{1 + m^2}{2}, \quad v_2 = \frac{1 - m^2}{4}. \quad (91)$$

Substituting (91) into the constraint conditions (14), the parameters can be determined as

$$k = 0, \quad \omega = a(1 - m^2), \quad \lambda^2 = \frac{nb(m^2 - 1)}{am^2(m^2 + 3)}, \quad c^2 = \frac{a(1 - m^2)}{4}. \quad (92)$$

then the complex wave solution to the Eq. (1) is

$$u = \frac{4}{n}\tanh^{-1}\left(\frac{\tan(\xi)}{nc(\eta, m) \pm sc(\eta, m)}\right). \quad (93)$$
**Case 21.** From Table 1, choosing $U = \text{sc}(\zeta, k)$ and $V = \text{sn}(\eta, m) \pm \text{cs}(\eta, m)$, $i^2 = -1$, respectively, and then from (13), we have

\[
\frac{\omega}{4c^2} = 1 - k^2, \quad \mu_1 = 2 - k^2, \quad \nu_1 = 1, \quad \frac{\omega}{4a} = \frac{m^2}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}.
\]

(94)

Substituting (94) into the constraint conditions (14), the parameters can be determined as

\[
k = 0, \quad \omega = am^2, \quad \lambda^2 = \frac{nbm^2}{a(4 - m^2)}, \quad c^2 = \frac{am^2}{4}.
\]

(95)

then the complex wave solution to the Eq. (1) is

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\tan(\zeta)}{\text{sn}(\eta, m) \pm \text{cs}(\eta, m)} \right).
\]

(96)

when $m \to 1$, the solution (96) turn to be another complex wave solution

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\tan(\zeta)}{\text{tanh}(\eta) \pm \text{csch}(\eta)} \right).
\]

(97)

**Case 22.** From Table 1, choosing $U = \text{cs}(\zeta, k)$ and $V = \text{sn}(\eta, m) \pm \text{cs}(\eta, m)$, respectively, and then from (13), we have

\[
\frac{\omega}{4c^2} = 1, \quad \mu_1 = 2 - k^2, \quad \nu_1 = 1 - k^2, \quad \frac{\omega}{4a} = \frac{1}{4}, \quad \mu_2 = \frac{1 - 2m^2}{2}, \quad \nu_2 = \frac{1}{4}.
\]

(98)

Substituting (98) into the constraint conditions (14), the parameters can be determined as

\[
k = 0, \quad \omega = a, \quad \lambda^2 = \frac{nb}{3am^2}, \quad c^2 = \frac{a}{4}.
\]

(99)

then the complex wave solution to the Eq. (1) is

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\cot(\zeta)}{\text{sn}(\eta, m) \pm \text{cs}(\eta, m)} \right).
\]

(100)

when $m \to 1$, the solution (100) turn to be another complex wave solution

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\cot(\zeta)}{\text{coth}(\eta) \pm \text{csch}(\eta)} \right).
\]

(101)

**Case 23.** From Table 1, choosing $U = \text{cs}(\zeta, k)$ and $V = \text{nc}(\eta, m) \pm \text{sc}(\eta, m)$, respectively, and then from (13), we have

\[
\frac{\omega}{4c^2} = 1, \quad \mu_1 = 2 - k^2, \quad \nu_1 = 1 - k^2, \quad \frac{\omega}{4a} = 1 - \frac{m^2}{4}, \quad \mu_2 = \frac{1 + m^2}{2}, \quad \nu_2 = \frac{1 - m^2}{4}.
\]

(102)

Substituting (102) into the constraint conditions (14), the parameters can be determined as

\[
k = 0, \quad \omega = a(1 - m^2), \quad \lambda^2 = \frac{nb(m^2 - 1)}{am^2(m^2 + 3)}, \quad c^2 = \frac{a(1 - m^2)}{4}.
\]

(103)

then the complex wave solution to the Eq. (1) is

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\cot(\zeta)}{\text{nc}(\eta, m) \pm \text{sc}(\eta, m)} \right).
\]

(104)

**Case 24.** From Table 1, choosing $U = \text{cs}(\zeta, k)$ and $V = \text{ns}(\eta, m) \pm \text{ds}(\eta, m)$, respectively, and then from (13), we have

\[
\frac{\omega}{4c^2} = 1, \quad \mu_1 = 2 - k^2, \quad \nu_1 = 1 - k^2, \quad \frac{\omega}{4a} = \frac{1}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}.
\]

(105)

Substituting (105) into the constraint conditions (14), the parameters can be determined as

\[
k = \sqrt{1 - m^2}, \quad \omega = a, \quad \lambda^2 = \frac{4nb}{3a(5 - m^2)}, \quad c^2 = \frac{a}{4}.
\]

(106)

then the complex wave solution to the Eq. (1) is
The profiles of (107) are shown in Fig. 1(1–7) and (1–8).

When \( m \rightarrow 0 \), the solution (107) turn to be another complex wave solution

\[
u = \pm \frac{4}{n} \tanh^{-1} \left( \frac{1}{2} \tanh(\xi) \sin(\eta) \right).
\]  

(108)

**Case 25.** From Table 1, choosing \( U = \cs(\xi, k) \) and \( V = \sn(\eta, m) \pm \ics(\eta, m) \), \( \iota^2 = -1 \), respectively, and then from (13), we have

\[
\frac{\omega}{4c^2} = 1, \quad \mu_1 = 2 - k^2, \quad \nu_1 = 1 - k^2, \quad \frac{\omega}{4a} = \frac{m^2}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}.
\]  

(109)

Substituting (109) into the constraint conditions (14), the parameters can be determined as

\[k = 0, \quad \omega = am^2, \quad \lambda^2 = \frac{m^2 nb}{a(4 - m^2)}, \quad c^2 = \frac{am^2}{4},\]

(110)

then the complex wave solution to the Eq. (1) is

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\cot(\xi)}{\sn(\eta, m) \pm \ics(\eta, m)} \right),
\]  

(111)

when \( m \rightarrow 1 \), the solution (111) turn to be another complex wave solution

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\cot(\xi)}{\tanh(\eta) \pm \icsch(\eta)} \right).
\]  

(112)

**Case 26.** From Table 1, choosing \( U = \ns(\xi, k) \pm \cs(\xi, k) \) and \( V = \nc(\eta, m) \pm \sc(\eta, m) \), respectively, and then from (13), we have

\[
\frac{\omega}{4c^2} = \frac{1}{4}, \quad \mu_1 = \frac{1}{4} - \frac{k^2}{2}, \quad \nu_1 = \frac{1}{4}, \quad \frac{\omega}{4a} = \frac{1 - m^2}{4}, \quad \mu_2 = \frac{m^2 + 1}{2}, \quad \nu_2 = \frac{1 - m^2}{4}.
\]  

(113)

Substituting (113) into the constraint conditions (14), the parameters can be determined as

\[k^2 = \frac{nb + a^2 m^4 - m^2 nb}{a^2 m^2 (m^2 - 1)}, \quad \omega = a (1 - m^2), \quad c^2 = a (1 - m^2),\]

(114)

then the complex wave solution to the Eq. (1) is

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\ns(\xi, k) \pm \cs(\xi, k)}{\nc(\eta, m) \pm \sc(\eta, m)} \right).
\]  

(115)

**Case 27.** From Table 1, choosing \( U = \ns(\xi, k) \pm \cs(\xi, k) \) and \( V = \sn(\eta, m) \pm \ics(\eta, m) \), \( \iota^2 = -1 \) respectively, and then from (13), we have

\[
\frac{\omega}{4c^2} = \frac{1}{4}, \quad \mu_1 = \frac{1}{4} - \frac{k^2}{2}, \quad \nu_1 = \frac{1}{4}, \quad \frac{\omega}{4a} = \frac{m^2}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}.
\]  

(116)

Substituting (116) into the constraint conditions (14), the parameters can be determined as

\[k^2 = \frac{am^2 \lambda^2 + m^2 nb - a^2 \lambda^2}{a^2 m^2 (m^2 - 1)}, \quad \omega = am^2, \quad c^2 = am^2,
\]  

(117)

then the complex wave solution to the Eq. (1) is

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\ns(\xi, k) \pm \cs(\xi, k)}{\sn(\eta, m) \pm \ics(\eta, m)} \right).
\]  

(118)

**Case 28.** From Table 1, choosing \( U = \nc(\xi, k) \pm \sc(\xi, k) \) and \( V = \nc(\eta, m) \pm \sc(\eta, m) \) respectively, and then from (13), we have

\[
\frac{\omega}{4c^2} = \frac{1 - k^2}{4}, \quad \mu_1 = \frac{1}{2} + \frac{k^2}{2}, \quad \nu_1 = \frac{1 - k^2}{4}, \quad \frac{\omega}{4a} = \frac{1 - m^2}{4}, \quad \mu_2 = \frac{1 + m^2}{2}, \quad \nu_2 = \frac{1 - m^2}{4}.
\]  

(119)

Substituting (119) into the constraint conditions (14), the parameters can be determined as
\[ \omega = a(1 - m^2), \quad \lambda^2 = \frac{nb(k^2 - 1)(1 - m^2)}{a(k^2 - m^2)}, \quad c^2 = \frac{a(1 - m^2)}{1 - k^2}, \] \hspace{1cm} (120)

and

\[ k = 1, \quad m = 1, \quad \omega = 0, \quad \lambda^2 = \frac{-nb^2}{(c^2 - a)^2}, \] \hspace{1cm} (121)

then the complex wave solutions to the Eq. (1) are

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\text{nc}(\xi, k) \pm \text{sc}(\xi, k)}{\text{nc}(\eta, m) \pm \text{sc}(\eta, m)} \right), \] \hspace{1cm} (122)

and

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\cosh(\xi) \pm \sinh(\xi)}{\cosh(\eta) \pm \sinh(\eta)} \right), \] \hspace{1cm} (123)

when \( k \to 0 \), the solution (122) turn to be another complex wave solution

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\sec(\xi) \pm \tan(\xi)}{\sec(\eta) \pm \tan(\eta)} \right), \] \hspace{1cm} (124)

when \( m \to 0 \), the solution (122) turn to be another complex wave solution

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\text{nc}(\xi, k) \pm \text{sc}(\xi, k)}{\sec(\eta) \pm \tan(\eta)} \right). \] \hspace{1cm} (125)

**Case 29.** From Table 1, choosing \( U = \text{nc}(\xi, k) \pm \text{sc}(\xi, k) \) and \( V = \text{sn}(\eta, m) \pm \text{cs}(\eta, m) \), \( \tilde{t}^2 = -1 \), respectively, and then from (13), we have

\[ \frac{\omega}{4c^2} = \frac{1 - k^2}{4}, \quad \mu_1 = \frac{1 + k^2}{2}, \quad \nu_1 = \frac{1 - k^2}{4}, \quad \omega = \frac{m^2}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}. \] \hspace{1cm} (126)

Substituting (126) into the constraint conditions (14), the parameters can be determined as

\[ \omega = am^2, \quad \lambda^2 = \frac{m^2nb(k^2 - 1)}{a(k^2m^4 - (1 - k^2)(1 + m^2))}, \quad c^2 = \frac{am^2}{1 - k^2}, \] \hspace{1cm} (127)

and

\[ k = 1, \quad m = 0, \quad \omega = 0, \quad \lambda^2 = \frac{nb^2}{a^2 - c^2}, \] \hspace{1cm} (128)

then the complex wave solutions to the Eq. (1) are

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\text{nc}(\xi, k) \pm \text{sc}(\xi, k)}{\text{sn}(\eta, m) \pm \text{cs}(\eta, m)} \right), \] \hspace{1cm} (129)

and

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\cosh(\xi) \pm \sinh(\xi)}{\sin(\eta) \pm i \cot(\eta)} \right), \] \hspace{1cm} (130)

when \( k \to 0 \), the solution (129) turn to be another complex wave solution

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\sec(\xi) \pm \tan(\xi)}{\text{sn}(\eta, m) \pm \text{cs}(\eta, m)} \right), \] \hspace{1cm} (131)

when \( m \to 1 \), the solution (129) turn to be another complex wave solution

\[ u = \frac{4}{n} \tanh^{-1} \left( \frac{\text{nc}(\xi, k) \pm \text{sc}(\xi, k)}{\tanh(\eta) \pm i \text{csch}(\eta)} \right). \] \hspace{1cm} (132)

**Case 30.** From Table 1, choosing \( U = \text{ns}(\xi, k) \pm \text{ds}(\xi, k) \) and \( V = \text{sn}(\eta, m) \pm \text{cs}(\eta, m) \), \( \tilde{t}^2 = -1 \), respectively, and then from (13), we have
\[
\frac{\omega}{4c^2} = \frac{1}{4}, \quad \mu_1 = \frac{k^2 - 2}{2}, \quad \omega_1 = \frac{k^2}{4}, \quad \omega_4 = \frac{m^2}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}.
\]

Substituting (133) into the constraint conditions (14), the parameters can be determined as

\[
k = 1, \quad \omega = am^2, \quad \chi^2 = \frac{m^2nb}{a(1-m^2)}, \quad n^2 = am^2,
\]

then the complex wave solution to the Eq. (1) is

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\coth(\zeta \pm \csc(\zeta))}{\sin(\eta, m) \pm \csc(\eta, m)} \right).
\]

**Case 31.** From Table 1, choosing \(U = sn(\zeta, k) \pm ics(\zeta, k)\) and \(V = sn(\eta, m) \pm ics(\eta, m)\), \(i^2 = -1\), respectively, and then from (13), we have

\[
\frac{\omega}{4c^2} = \frac{k^2}{4}, \quad \mu_1 = \frac{k^2 - 2}{2}, \quad \omega_1 = \frac{k^2}{4}, \quad \omega_4 = \frac{m^2}{4}, \quad \mu_2 = \frac{m^2 - 2}{2}, \quad \nu_2 = \frac{m^2}{4}.
\]

Substituting (136) into the constraint conditions (14), the parameters can be determined as

\[
\omega = am^2, \quad \chi^2 = \frac{k^2m^2nb}{a(k^2 - m^2)}, \quad n^2 = \frac{am^2}{k^2},
\]

and

\[
m = 0, \quad k = 0, \quad \omega = 0, \quad \chi^2 = \frac{nb^2}{(c^2 - a)^2},
\]

then the complex wave solutions to the Eq. (1) are

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{sn(\zeta, k) \pm ics(\zeta, k)}{\sin(\eta, m) \pm \csc(\eta, m)} \right),
\]

and

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\sin(\zeta \pm i\cot(\zeta))}{\sin(\eta \pm i\cot(\eta))} \right),
\]

when \(k \to 1\), the solution (139) turn to be another complex wave solution

\[
u = \frac{4}{n} \tanh^{-1} \left( \frac{\tanh(\zeta \pm i\csc(\zeta))}{\sin(\eta, m) \pm \csc(\eta, m)} \right).
\]

**4. Conclusion**

In this paper, we obtained many exact solutions of a generalized sinh–Gordon equation, these solutions involve combinations of the arctanh function, Jacobi elliptic functions, hyperbolic functions and trigonometric functions. Compare with Refs. [8,9], most solutions of our results are entirely new.

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**References**